Section 1

1. Let (a_n) be a cauchy sequence of real numbers. Suppose there is a subsequence (a_{k_n}) such that $a_{k_n} \to a$. Prove that $a_n \to a$.
2. Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \in \mathbb{Z}$ for all $x \in \mathbb{R}$.
3. Suppose f is a function satisfying $|f(x) - f(y)| \leq |x - y|^2$ for all $x,\ y \in \mathbb{R}$. Show that f is constant.

Solution: (1) We have (a_n) is cauchy and $a_{k_n} \to a$, so for each $\epsilon > 0$ there exist a $M \in \mathbb{N}$ such that

$$|a_n - a_m| < \frac{\epsilon}{2}, \quad |a_{n_k} - a| < \frac{\epsilon}{2}, \ \forall \ \ n, m, n_k > M.$$

Now take $n_k > M$ then we have

$$|a_n - a| = |a_n - a_{n_k} + a_{n_k} - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \ \ \forall \ n > M.$$

Hence $a_n \to a$.

(2) f is continuous function so $f(\mathbb{R})$ is a conneted set in $\mathbb{R}$. We are given $f(\mathbb{R}) \subset \mathbb{Z}$ and only connected subset of $\mathbb{Z}$ are singleton sets. So $f(\mathbb{R}) = n$ for some $n \in \mathbb{Z}$. So all the continuous function satisfying $f(R) \subset \mathbb{Z}$ is given by $\{f(x) = n, \ \forall \ x \in \mathbb{R}\}_{n \in \mathbb{Z}}$.

(3) We have for any $x_0 \in \mathbb{R}$

$$|f'(x_0)| = \lim_{h \to 0} \left| \frac{1}{h}\big(f(x_0 + h) - f(x_0)\big) \right| \leq \lim_{h \to 0} \frac{h^2}{h} = 0.$$

So we have $f' = 0$ on $\mathbb{R}$ i.e f is a constant function. □

Section 2

1. Prove that cauchy sequence are convergent.
2. Let (a_n) be a sequence of real numbers. Let $b_n = a_n + |a_n|$ and $c_n = |a_n| - a_n$ for all $n \geq 1$. Prove that $\sum_n a_n$ converges absolutely if and only if $\sum_n a_n$ and $\sum_n b_n$ converge.
3. Prove that a continuous function on $[a, b]$ is uniformly continuous.
4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with IVP and $x \in \mathbb{R}$. Suppose $\lim f(x_n) = f(x)$ for any sequence $x_n \to x$ with $(f(x_n))$ is a constant sequence. Prove that f is continuous at x.
5. Let $f : (0, 1) \to \mathbb{R}$ is a differentiable function having a local maximum at $a \in (0, 1)$. Prove that $f'(a) = 0$.
6. Let $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such $f(x + y) = f(x) + f(y)$ for any $x,\ y \in [0, 1]$. Prove that there exists $a \in \mathbb{R}$ such that $f(x) = ax$ for all $x \in [0, 1]$.

Solution: (1) See 3.11 Theorem of W. Rudin (principles of mathematical analysis).

(2) Let assume $\sum_n a_n$ converges absolutely then $\sum_n b_n$ converges as $|b_n| \leq 2|a_n|$.

Now $\sum_{n=1}^N a_n = \sum_{n=1}^N b_n - \sum_{n=1}^N |a_n|$ as $b_n = a_n + |a_n|$. Both $\lim_{N\to\infty} \sum_{n=1}^N b_n$ and $\lim_{N\to\infty} \sum_{n=1}^N |a_n|$ exist so $\lim_{N\to\infty} \sum_{n=1}^N a_n$ exist i.e $\sum_n a_n$ converges.
Now assume $\sum_n a_n$ and $\sum_n b_n$ converge the using $\sum_{n=1}^N |a_n| = \sum_{n=1}^N b_n - \sum_{n=1}^N a_n$ we get the absolutely convergence of $\sum_n a_n$.

(3) See 4.19 Theorem of W. Rudin (principles of mathematical analysis) and use the fact that $[a, b]$ is a compact set in $\mathbb{R}$.

(4) Let f is not continuous at x then there exist a sequence (u_n) such that $u_n \to x$ but $f(u_n) \nrightarrow f(x)$. So for each $\epsilon > 0$ there exist infinitely many terms say $(u_{n_k})_k$ of (u_n) such that $|f(u_{n_k}) - f(x)| > \epsilon \;\; \forall\, n_k$ i.e $f(u_{n_k}) > f(x)+\epsilon$ or $f(u_{n_k}) < f(x)-\epsilon$ will happen for infinitely n_k, W.l.o.g assume that $f(v_j) > f(x)+\epsilon$ holds for a subsequence say $(v_j)_j$ of (u_{n_k}) $(v_j = u_{n_{k_j}})$. $f(x) < f(x) + \epsilon < f(v_j)$ so by IVP we get there exist $z_j \in (x, v_j)$ *or* (v_j, x) such that $f(z_j) = f(x) + \epsilon$ Now we have $z_j \to x$ but $\lim_{j\to\infty} = f(x) + \epsilon$ as $f(z_j)$ is a constant sequence namely $f(x) = \epsilon$. which give the contradiction to the condition given in the problem. So f is continuous at x.

(5) f is differentiable function so we have $f'(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}$.
Let assume $f'(a) > 0$ then $\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} > 0$ which will imply that there is a $\delta > 0$ such that $f(a + h) > f(a)\; \forall\; 0 < h < \delta$. This contradict the fact that f has local maximum at a.
Similar if $f'(a) < 0$ using $f'(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}$ we will get again a contradiction to the fact that f has local maximum at a. So $f'(a) = 0$.

(6) We have $f(0 + 0) = f(0) + f(0)$ which give $f(0) = 0$. Now

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = f'(0), \quad f(x + h) = f(x) + f(h).$$

Let $a = f'(0)$ then $f'(x) = a$ together with $f(0) = 0$ will give $f(x) = ax$. $\square$

Section 3

1. Let (a_n) be a sequence of real numbers.
(a) If c is a limit point of (a_n). Prove that there is a subsequence (a_{k_n}) such that $a_{k_n} \to c$ and $\underline{\lim}\, a_n \le c \le \overline{\lim}\, a_n$.
(b) Prove that $a_n \to \infty$ if and only if $\underline{\lim}\, a_n = \infty$.

2. (a) $f : \mathbb{R} \to \mathbb{R}$ be a function with IVP. Can f have simple discontinuties? Justify your answer.
(b) $f : (a, b) \to \mathbb{R}$ be a strictly increasing continuous function. Prove that there are extended real numbers A and B and a continuous function $\phi : (A, B) \to (a, b)$ such that $\phi(f(x)) = x$ for all $x \in (a, b)$.

3. (a) Let $f : [a, b] \to \mathbb{R}$ be a differentiable function and $A > 0$ such that $f(a) = 0$ and $|f'(x)| \le A|f(x)|$ for all $x \in [a, b]$ Prove that $f = 0$ on $[a, b]$.
(b) Prove Taylors theorem: Let $f : [0, 1] \to \mathbb{R}$ be a function such that f' exists and continuous on $[0, 1]$ and f'' exists on $(0, 1)$. Prove that there is a $t \in (0, 1)$ such that $f(1) = f(0) + f'(0) + \frac{f''(t)}{2}$.

Solution 1. (a) Since c is a limit point of (a_n) then for each $k \in \mathbb{N}$ there exist $M_k \in \mathbb{N}$ such that $|a_{n_k} - c| < \frac{1}{2^k} \;\; \forall\;\; n > M_k$. Now we can chose a subsequence (a_{n_k}) such that $n_k > M_k$ and $|a_{n_k} - c| < \frac{1}{2^k}$. Now it is easy to see that $\lim_{n_k\to\infty} a_{n_k} = c$.

Now let E be the set of all limit points of (a_n) then $\underline{\lim}\, a_n = \inf\ E$ and $\overline{\lim} = \sup\ E$ (see 3.16 Definition of W. Rudin). So we have $\underline{\lim}\, a_n \leq c \leq \overline{\lim}\, a_n$ as $c \in E$.
(b) Also we can write $\underline{\lim}\, a_n = \sup_n \inf_{k\geq n} a_n = \lim_{n\to\infty} \inf_{k\geq n} a_n$. Let $\underline{\lim}\, a_n = \infty$ then we get for each $M > 0$ there exit $N \in \mathbb{N}$ such that $\inf_{k\geq n} > M \quad \forall\ n > N$ thats give $a_k > M \quad \forall\ k > N$ this imply $a_n \to \infty$.
Now assume $a_n \to \infty$ then for each $M > 0$ there exist $N \in \mathbb{N}$ such that $a_n > M \quad \forall\, k \geq N$. This will give $\inf_{k\geq N} a_k > M$ which imply $\sup_n \inf_{k\geq n} > M \quad \forall\ n \geq N$. So we get $\underline{\lim}\, a_n = \infty$.

2.(a) Define the function f as following

$$f(x) = \begin{cases} -x & \text{if } \ x \in (-1, 0) \\ \frac{1}{2} & \text{if } \ x = 0 \\ x & \text{if } \ x \in (0, 1) \end{cases}$$

Then f be a function with IVP but has simple discontinuty at $x = 0$.
(b) Define $A = \lim_{x\to a^+} f(x)$ and $B = \lim_{x\to b^-} f(x)$. Since f is a increasing function so it has at most countable discontinuties. Let $D = \{x_1, x_2, \cdots, x_n, \cdots\}$ be the set of discontinuities of f. Then we can write $(a, b) = \cup_{n\geq 1}(u_n, v_n) \cup D, \quad v_n < x_n < u_{n+1}$ such that f is continuous at each (u_n, v_n) and this will give $f(u_n, v_n) = (f(u_n), f(v_n))$. Since f is stictly increasing (one-one) $f^{-1} : f(a, b) \to (a, b)$ exist. first we will prove f^{-1} is continuous and this will be done by showing $f(C)$ is open whenever C is open in (a, b). Let $(r_1, r_2) \subset (a, b)$ if $(r_1, r_2) \subset (u_n, v_n)$ for some n then $f(r_1, r_2) = (f(r_1), f(r_2))$ is open in $f(a, b)$. W.l.g.n assume $(r_1, r_2) \subset (u_n, v_n) \cup (u_{n+1}, v_{n+1}) \cup \{r_n\}$ then $f(r_1, r_2) = (f(u_n), f(v_{n+1})) \cap f(a, b)$ is open in $f(a, b)$. We can alway find a continuous function $g : (A, B) \setminus f(a, b) \to \mathbb{R}$ with the fact that $a \leq g(x) \leq b\ \forall\ x \in (A, B) \setminus f(a, b)$. Now we can define $\phi : (A, B) \to (a, b)$ as

$$\phi(x) = \begin{cases} f^{-1}(x) & \text{if } \ x \in f(a, b) \\ g(x) & \text{if } \ x \in (A, B) \setminus f(a, b) \end{cases}$$

We have continuous function $\phi : (A, B) \to \ (a, b)$ such that $\phi(f(x)) = x$ for all $x \in (a, b)$.

3. (a) Let $x_0 = \inf\limits_{x\in[a,b]} \{x : f'(x) \neq 0\}$. If $x_0 = a$ then $f' = 0$ and we are done. So assume $a < x_0 \leq b$. Now mean value theorem will give

$$f(x_0) = f(x_0) - f(a) = f'(c)(x_0 - a) \quad a < c < x_0$$

From above we get $0 < \frac{|f'(x_0)|}{A} \leq |f(x_0)| = |x_0 - a||f'(c)|$ which give $f'(c) \neq 0$. But definition of x_0 and $a < c < x_0$ will give $f'(c) = 0$. So we got a contradiction which will imply $x_0 = a$ i.e $f' = 0$ and $f(a) = 0$ give $f = 0$.
(b) See the 5.15 Theorem of W. Rudin with $[a, b] = [0, 1], \quad n = 2, \quad \alpha = 0$ and $\beta = 1$. $\square$